Homework I
Due Date: 17/03/2022

Exercise 1 (1 point). Let (u, v) be the C^2 solution to the boundary problem of the following 3D coupled elliptic equations:

$$\begin{cases} -\Delta u - (1 - u^2 - v^2)u = 0, & \text{in } B(0,1), \\ -\Delta v - (1 - u^2 - v^2)v = 0, & \text{in } B(0,1), \\ u(x) = 0 \text{ and } v(x) = 0, & \text{on } \partial B(0,1). \end{cases}$$

(i) Let $w(x) = u^2(x) + v^2(x)$ and then compute the equation which w satisfy.
(ii)Show that $\max\limits_{x \in \bar{B}(0,1)} \left(u^2(x) + v^2(x)\right) \leq 1$.

Exercise 2 (2 points). Let $\mathcal{F}$ be the set of functions $f(x, y)$ that are twice continuously differentiable for $x \geq 1$, $y \geq 1$ and that satisfy the following two equations:

$$x\partial_x f + y\partial_y f = xy \log(xy) \quad \text{and} \quad x^2\partial_x^2 f + y^2\partial_y^2 f = xy. \tag{1}$$

(i) Deduce the formula of $\partial_{xy} f$ for $f \in \mathcal{F}$ from (1).
(ii) For each $f \in \mathcal{F}$, we let

$$m(f) = \min_{s \geq 1} \left\{ f(s+1, s+1) - f(s+1, s) - f(s, s+1) + f(s, s) \right\}.$$

Determine $m(f)$, and show that it is independent of the choice of f.

Exercise 3 (2 points). Consider the 1D heat equation

$$\partial_t u(t, x) = \partial_x^2 u(t, x), \quad (t, x) \in \mathbb{R}^+ \times (0, 1),$$

with $u(t, 0) = u(t, 1) = 0$ and $u(0, x) = 4x(1 - x)$.
(i) Show that $0 < u(t, x) < 1$ for all $(t, x) \in \mathbb{R}^+ \times (0, 1)$.
(ii) Show that $u(t, x) = u(t, 1 - x)$ for all $t \geq 0$ and $0 \leq x \leq 1$.
(iii) Show that $\int_0^1 u^2(t, x)$ is a strictly decreasing function of t.

Exercise 4 (1.5 points). (i) Solve the 1D heat equation with constant dissipation

$$\begin{cases} \partial_t u = \partial_x^2 u + u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) = \phi(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

(ii) Solve the 1D heat equation with variable dissipation

$$\begin{cases} \partial_t u = \partial_x^2 u + t^2 u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) = \phi(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

(iii) Solve the 1D heat equation with convection

$$\begin{cases} \partial_t u = \partial_x^2 u + \partial_x u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \\ u(0, x) = \phi(x), & \text{for } x \in \mathbb{R}. \end{cases}$$

Exercise 5 (2 points) Consider the following problem with a Robin boundary condition:

$$\begin{cases} \partial_t u = \partial_x^2 u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(0, x) = x, & \text{for } x \in \mathbb{R}^+, \\ \partial_x u(t, 0) - 2u(t, 0) = 0, & \text{for } t \in \mathbb{R}^+. \end{cases} \tag{2}$$

The purpose of this exercise is to verify the solution formula for (2). Let $f(x) = x$ for $x > 0$, let $f(x) = 1 + x - e^{2x}$ for $x < 0$, and let

$$v(t,x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4t}} f(y) \mathrm{d}y.$$

(i) Let $w = \partial_x v - 2v$. What PDE and initial condition does $w(t,x)$ satisfy for $x \in \mathbb{R}$.
(ii) Show that $f'(x) - 2f(x)$ is an odd function for $x \neq 0$.
(iii) Show that $w(t,x)$ is an odd function of x.
(iv) Deduce that $v(t,x)$ satisfies (2) for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$.

Exercise 6 (1.5 points) Using the method of Exercise 5 to solve the following generalize Robin problem:

$$\begin{cases} \partial_t u = \partial_x^2 u, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(0,x) = \phi(x), & \text{for } x \in \mathbb{R}^+, \\ \partial_x u(t,0) - hu(t,0) = 0, & \text{for } t \in \mathbb{R}^+, \end{cases} \tag{3}$$

where $h \in \mathbb{R}^+$ is a constant.